Forschungszentrum Telekommunikation Wien

[Telecommunications Research Center Vienna]

Theory and Design of Turbo and Related Codes

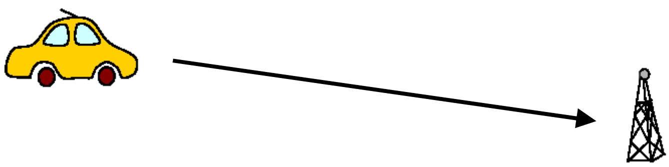
Lecture 3

Jossy Sayir & Gottfried Lechner

Kplus

Kompetenzzentren-Programm

The Channel



- The channel is the link between the transmitter and the receiver.
- Signals sent by the transmitter will usually be corrupted by the channel.
- The corruption of the signal has some random source.
- We use channel models that approximate real channels.

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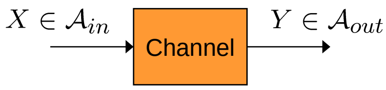
The Channel



The channel maps a symbol of the input alphabet to a symbol of the output alphabet (randomly).

This mapping can be described by the probability of a symbol at the output, given that a symbol at the input has been transmitted.

$$P(Y = y|X = x)$$

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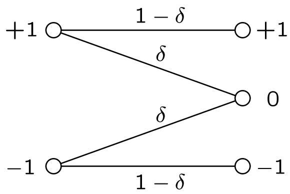
Channels with Discrete Output

Binary Erasure Channel
BEC

$X \in \mathcal{A}_{in} = \{+1, -1\}$

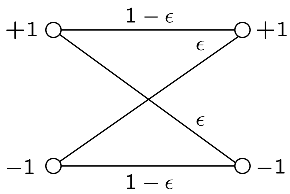
$Y \in \mathcal{A}_{out} = \{+1, 0, -1\}$



Binary Symmetric Channel
BSC

$X \in \mathcal{A}_{in} = \{+1, -1\}$

$Y \in \mathcal{A}_{out} = \{+1, -1\}$



Does the decoding algorithm of lecture 1 also work for a BSC?

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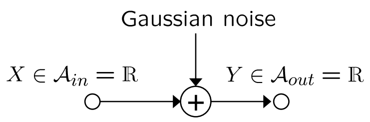
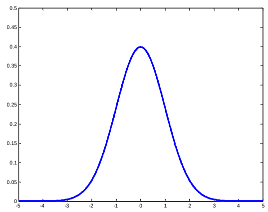
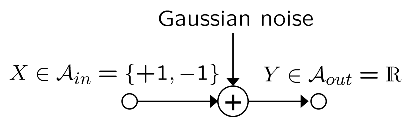
Channels with Continuous Output

Additive White Gaussian Noise
AWGN





Binary Input AWGN
BIAWGN



$$P(Y = y | X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-x)^2}{2\sigma^2}}$$

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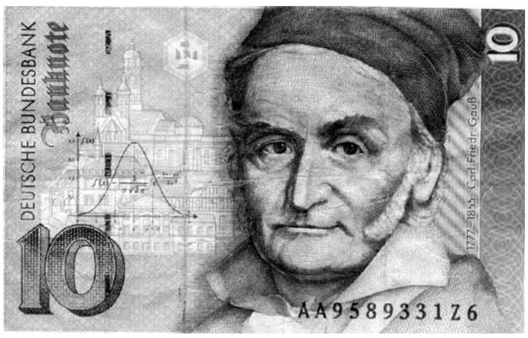
Carl Friedrich Gauss 1777-1855



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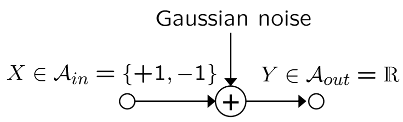
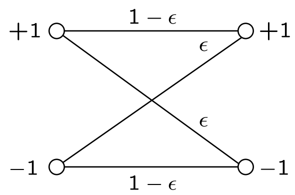
BIAWGN vs. BSC

A BIAWGN channel with "hard decision" becomes a BSC.



$$P(Y = y|X = x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-x)^2}{2\sigma^2}}$$



$$\epsilon = P(Y > 0|X = -1) = \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y+1)^2}{2\sigma^2}} dy$$

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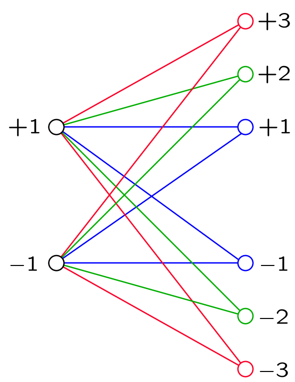
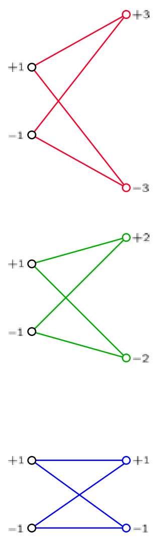
Higher output Alphabet





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Channels - Summary

- We use only binary input channels in this course.
- Channels are memoryless.
- Fully described by their transition probabilities.
- All channels satisfy a symmetry property

$$P(Y = y|X = x) = P(Y = -y|X = -x)$$

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Abstraction of Channels

In the end we are only interested which binary digit was transmitted.

Therefore, we have to compare probabilities of +1 or -1 conditioned on our observation.

$$P(X = +1|Y = y) \gtrless P(X = -1|Y = y)$$

$$P(X = +1|Y = y) - P(X = -1|Y = y) \gtrless 0$$

$$\frac{P(X = +1|Y = y)}{P(X = -1|Y = y)} \gtrless 1$$

$$\log \frac{P(X = +1|Y = y)}{P(X = -1|Y = y)} \gtrless 0$$

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Log-Likelihood Ratio

$$\mathsf{L}(X|Y=y) = \log \frac{P(X=+1|Y=y)}{P(X=-1|Y=y)}$$

$$= \log \frac{P(Y=y|X=+1)\cdot P(X=+1)}{P(Y=y|X=-1)\cdot P(X=-1)}$$

$$= \underbrace{\log \frac{P(Y=y|X=+1)}{P(Y=y|X=-1)}}_{\text{channel L-value}} + \underbrace{\log \frac{P(X=+1)}{P(X=-1)}}_{\text{a-priori L-value}}$$

$$\mathsf{L}_{X,intr}(y) = \mathsf{L}_{X,ch}(y) + \mathsf{L}_{X,apri}$$

Note: $\mathsf{L}_{X,intr}(y)$ is a only a function of y - not of x! We will omit the subscript X if it is clear from the context.

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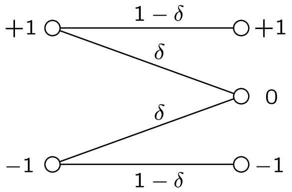
LLR of some Channels

BEC

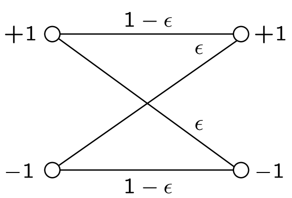


$$\mathsf{L}_{intr}(+1) = L_{apri} + \infty$$

$$\mathsf{L}_{intr}(0) = L_{apri} + 0$$

$$\mathsf{L}_{intr}(-1) = L_{apri} - \infty$$

BSC



$$\mathsf{L}_{intr}(+1) = L_{apri} + \log \frac{1-\epsilon}{\epsilon}$$

$$\mathsf{L}_{intr}(-1) = L_{apri} + \log \frac{\epsilon}{1-\epsilon}$$

Note the symmetry of the channels.

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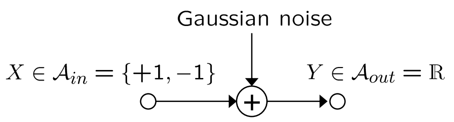
LLR of some Channels

BIAWGN



$$
\begin{aligned}
\mathsf{L}_{ch}(y) &= \log \frac{P(Y=y|X=+1)}{P(Y=y|X=-1)} \\
&= \log \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y-1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(y+1)^2}{2\sigma^2}}} \\
&= \frac{-(y-1)^2 + (y+1)^2}{2\sigma^2} = \frac{2y}{\sigma^2}
\end{aligned}
$$

The L-Value of the channel is a scaled version of the received value y.

$$\mathsf{L}_{intr}(y) = \frac{2y}{\sigma^2} + L_{apri}$$

The scaling determines the weighting between observation and a-priori information.

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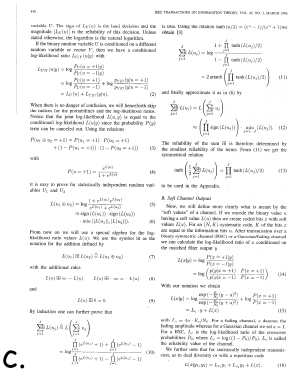
LLR Arithmetic

$$
\begin{aligned}
\sum_{j=1}^{J}\!\!\boxplus\, L(u_j) &= \log \frac{1 + \prod_{j=1}^{J} \tanh\left(L(u_j)/2\right)}{1 - \prod_{j=1}^{J} \tanh\left(L(u_j)/2\right)} \\
&= 2\,\mathrm{artanh}\left(\prod_{j=1}^{J} \tanh\left(L(u_j)/2\right)\right)
\end{aligned}
$$



Joachim Hagenauer (TU Munich)
A great advocate of LLR arithmetic.

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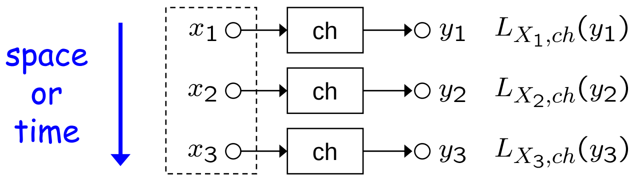
Calculating with LLR



$$\mathsf{L}(X_1|\underline{Y}=\underline{y}) = \log\frac{P(X_1=+1|\underline{Y}=\underline{y})}{P(X_1=-1|\underline{Y}=\underline{y})} = \log\frac{P(X_1=+1)}{P(X_1=-1)} + \log\frac{P(\underline{Y}=\underline{y}|X_1=+1)}{P(\underline{Y}=\underline{y}|X_1=-1)}$$

$$= \underbrace{\log\frac{P(X_1=+1)}{P(X_1=-1)}}_{\text{a-priori L-value}} + \underbrace{\log\frac{P(Y_1=y_1|X_1=+1)}{P(Y_1=y_1|X_1=-1)}}_{\text{channel L-value}} + \underbrace{\log\frac{P(\underline{Y}_{[1]}=\underline{y}_{[1]}|X_1=+1)}{P(\underline{Y}_{[1]}=\underline{y}_{[1]}|X_1=-1)}}_{\text{extrinsic L-value}}$$

$$\mathsf{L}_{X_1,app}(\underline{y}) = \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,ch}(y_1) + \mathsf{L}_{X_1,extr}(\underline{y}_{[1]})$$

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Note on Notation

For a vector $\underline{y}$ with its i'th element removed we write

$$\underline{y}_{[i]} = [y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n]$$

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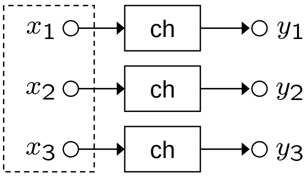
Repeat Code



$L_{X_1,ch}(y_1)$

$L_{X_2,ch}(y_2)$

$L_{X_3,ch}(y_3)$

$x_1 = x_2 = x_3$

$$\mathsf{L}_{X_1,extr}(\underline{y}_{[1]}) = \log \frac{P(\underline{Y}_{[1]} = \underline{y}_{[1]} | X_1 = +1)}{P(\underline{Y}_{[1]} = \underline{y}_{[1]} | X_1 = -1)}$$

$$= \log \frac{P(Y_2 = y_2, Y_3 = y_3 | X_2 = +1, X_3 = +1)}{P(Y_2 = y_2, Y_3 = y_3 | X_2 = -1, X_3 = -1)}$$

$$= \log \frac{P(Y_2 = y_2 | X_2 = +1)}{P(Y_2 = y_2 | X_2 = -1)} + \log \frac{P(Y_3 = y_3 | X_3 = +1)}{P(Y_3 = y_3 | X_3 = -1)}$$

$$\mathsf{L}_{X_1,extr}(\underline{y}_{[1]}) = \mathsf{L}_{X_2,ch}(y_2) + \mathsf{L}_{X_3,ch}(y_3)$$

$$\mathsf{L}_{X_1,app}(\underline{y}) = \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,ch}(y_1) + \mathsf{L}_{X_2,ch}(y_2) + \mathsf{L}_{X_3,ch}(y_3)$$

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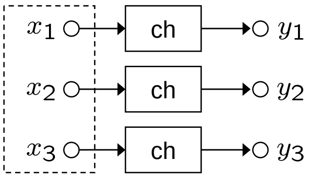
Single Parity-Check Code



$L_{X_1,ch}(y_1)$

$L_{X_2,ch}(y_2)$

$L_{X_3,ch}(y_3)$

$x_1 \cdot x_2 \cdot x_3 = +1$

$$\mathsf{L}(X_1 | \underline{Y} = \underline{y}) = \log \frac{P(X_1 = +1)}{P(X_1 = -1)} + \log \frac{P(Y_1 = y_1 | X_1 = +1)}{P(Y_1 = y_1 | X_1 = -1)} + \log \frac{P(\underline{Y}_{[1]} = \underline{y}_{[1]} | X_1 = +1)}{P(\underline{Y}_{[1]} = \underline{y}_{[1]} | X_1 = -1)}$$

$$= \log \frac{P(Y_1 = y_1 | X_1 = +1)}{P(Y_1 = y_1 | X_1 = -1)} + \log \frac{P(X_1 = +1 | \underline{Y}_{[1]} = \underline{y}_{[1]})}{P(X_1 = -1 | \underline{Y}_{[1]} = \underline{y}_{[1]})}$$

$$\mathsf{L}_{X_1,ch}(y_1) \qquad \mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,extr}(\underline{y}_{[1]})$$

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Gallager-Lemma

Consider a sequence of m independent binary digits in which the i'th digit is 1 with probability P_i. Then the probability that an even number of digits are 1 is

$$\frac{1+\prod_{i=1}^{m}(1-2P_i)}{2}$$

The probability that an odd number of digits are 1 is

$$1-\frac{1+\prod_{i=1}^{m}(1-2P_i)}{2}=\frac{1-\prod_{i=1}^{m}(1-2P_i)}{2}$$

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Single Parity-Check Code

$$\log\frac{P(X_1=+1|\underline{Y}_{[1]}=\underline{y}_{[1]})}{P(X_1=-1|\underline{Y}_{[1]}=\underline{y}_{[1]})}$$

$$P(X_1=-1|\underline{Y}_{[1]}=\underline{y}_{[1]}) = P(X_2{\cdot}X_3=-1|\underline{Y}_{[1]}=\underline{y}_{[1]})$$

$$=\frac{1-\prod_{i\neq 1}\left[1-2P(X_i=-1|\underline{Y}_{[1]}=\underline{y}_{[1]})\right]}{2}$$

$$1-2P(X_1=-1|\underline{Y}_{[1]}=\underline{y}_{[1]}) = \prod_{i\neq 1}\left[1-2P(X_i=-1|\underline{Y}_{[1]}=\underline{y}_{[1]})\right]$$

$$=\prod_{i\neq 1}\left[1-2P(X_i=-1|Y_i=y_i)\right]$$

$$\tanh\frac{L_{X_1,apri}+L_{X_1,extr}(\underline{y}_{[1]})}{2}=\prod_{i\neq 1}\tanh\frac{L_{X_i,intr}(y_i)}{2}$$

$$L_{X_1,apri}+L_{X_1,extr}(\underline{y}_{[1]}) = 2{\cdot}\tanh^{-1}\prod_{i\neq 1}\tanh\frac{L_{X_i,intr}(y_i)}{2}$$

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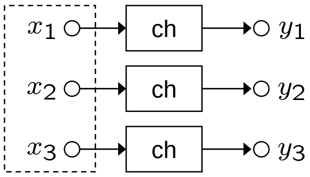
Single Parity-Check Code



$L_{X_1,ch}(y_1)$

$L_{X_2,ch}(y_2)$

$L_{X_3,ch}(y_3)$

$x_1 \cdot x_2 \cdot x_3 = +1$

$$\mathsf{L}(X_1|\underline{Y}=\underline{y}) = L_{X_1,ch}(y_1) + 2\cdot\tanh^{-1}\left[\prod_{i\neq 1}\tanh\frac{L_{X_i,intr}(y_i)}{2}\right]$$

$\mathsf{L}_{X_1,ch}(y_1)$ $\mathsf{L}_{X_1,apri} + \mathsf{L}_{X_1,extr}(\underline{y}_{[1]})$

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Calculating with LLR - Summary

If we receive independent noisy observations of the same binary random variable, we can add their channel L-values.

The L-value of a binary random variable that is the XOR of other random variables can be calculated using the tanh-rule.

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Single Parity-Check Code Example

Parity-Check code $x_1 \cdot x_2 \cdot x_3 = +1$

Assumption: no a-priori information ($L_{ch} = L_{intr}$)

L_{intr}=[+1.50 -2.00 +2.00]

$$\mathrm{L}(X_1|\underline{Y}=\underline{y}) = L_{X_1,ch}(y_1) + 2\cdot\tanh^{-1}\left[\prod_{i\neq 1} \tanh \frac{L_{X_i,intr}(y_i)}{2}\right]$$

L_{app}=[+0.18 -0.94 +0.94]

Decoder output: +1 -1 +1

This is not a codeword!!!

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Single Parity-Check Code Sequence Probability

$$P(\underline{X}=\underline{x}|\underline{Y}=\underline{y}) = P(\underline{Y}=\underline{y}|\underline{X}=\underline{x}) \cdot \frac{P(\underline{X}=\underline{x})}{P(\underline{Y}=\underline{y})}$$

$$= \frac{P(\underline{X}=\underline{x})}{P(\underline{Y}=\underline{y})} \cdot \prod_i P(Y_i=y_i|X_i=x_i)$$

$$= \frac{P(\underline{X}=\underline{x})}{P(\underline{Y}=\underline{y})} \cdot \prod_i P(X_i=x_i|Y_i=y_i) \cdot \frac{P(Y_i=y_i)}{P(X_i=x_i)}$$

$$P(X=+1|Y=y) = \frac{e^{L_{ch}(y)}}{1+e^{L_{ch}(y)}} = \frac{e^{\frac{L_{ch}(y)}{2}}}{e^{-\frac{L_{ch}(y)}{2}} + e^{\frac{L_{ch}(y)}{2}}}$$

$$P(X=-1|Y=y) = \frac{1}{1+e^{L_{ch}(y)}} = \frac{e^{-\frac{L_{ch}(y)}{2}}}{e^{-\frac{L_{ch}(y)}{2}} + e^{\frac{L_{ch}(y)}{2}}}$$

$$\mathrm{argmax}_{\underline{x}\in\mathcal{C}}\left\{P(\underline{X}=\underline{x}|\underline{Y}=\underline{y})\right\} = \mathrm{argmax}_{\underline{x}\in\mathcal{C}}\left\{\sum_i L_{X_i,ch}\cdot X_i\right\}$$

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Single Parity-Check Code
Sequence Probability - Example

L_{intr}=[+1.50 -2.00 +2.00]

$\underline{x}$	$\sum_i L_{X_i,intr} \cdot X_i$
+1 +1 +1	+1.50
+1 -1 -1	+1.50
-1 -1 +1	**+2.50**
-1 +1 -1	-5.50

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Summary Decoding strategies

- If the decoder uses a-priori information it is called maximum a-posteriori (MAP) decoder.
- If the decoder uses no a-priori information it is called maximum likelihood (ML) decoder.
- The decoder can be designed to find the most likely symbol or the most likely sequence. This is independent of the use of a-priori information!

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