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Forschungszentrum Telekommunikation Wien

Theory and Design of  
Turbo and Related Codes  
*Lecture 12*

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<http://userver.ftw.at/~jossy/turbo/index.html>

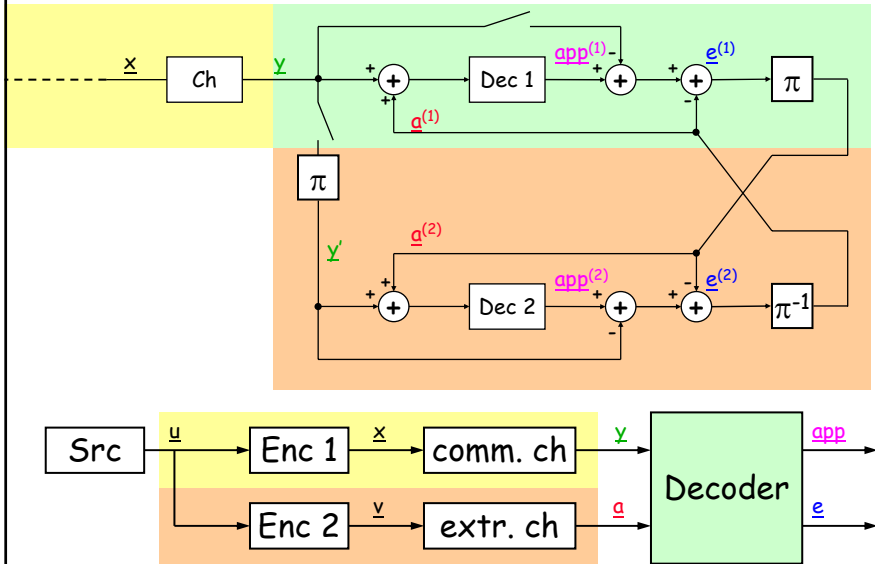


Last Lecture 

- We computed **information transfer functions** of the component decoders.
- We combined these functions to **EXIT charts**.
- We used these charts to predict the **convergence** behavior of the iterative decoder.
- We made some **assumptions** and/or **simplifications**.

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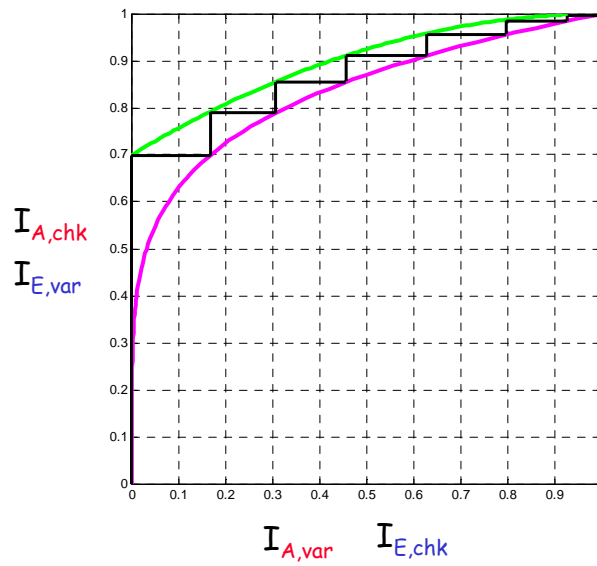
## Extrinsic Channel Model (from last lecture)



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## EXIT Chart of LDPC Code (from last lecture)



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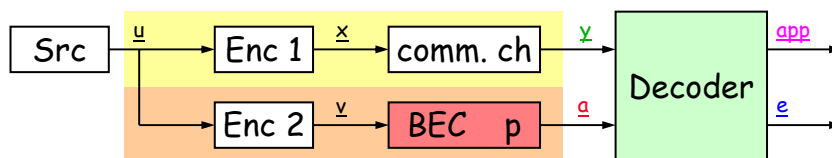
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## Summary of Assumptions (from last lecture)



- Messages received from the extrinsic channel are **independent observations**, which is only fulfilled if  $N \rightarrow \infty$
- We use **statistical quantities**, which are only correct if  $N \rightarrow \infty$
- We **model extrinsic messages** with an extrinsic channel. This can only be done exact for the BEC. The Gaussian assumption is an approximation.

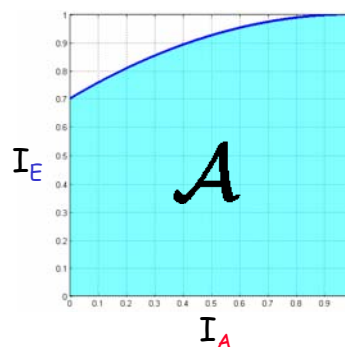
## Area Property



$$m = |v|$$

$$I_A = \frac{1}{m} \sum_{i=1}^m I(V_i; A_i) = 1 - p$$

$$\mathcal{A} = \int_0^1 I_E(I_A) dI_A = \int_0^1 I_E(p) dp$$



## Derivation of Area Property 1



$$\begin{aligned}
 I_E &= \frac{1}{m} \sum_{i=1}^m \underbrace{I(V_i; \underline{Y} \underline{A}_i)} \\
 &\quad \underbrace{1}_{\nearrow} \left[ \underbrace{H(V_i) - H(V_i | \underline{Y} \underline{A}_i)}_{\sum_{\underline{a}_i} P(\underline{a}_i) \cdot H(V_i | \underline{Y}, \underline{A}_i = \underline{a}_i)} \right] \quad \mathcal{S} = \{b = 1 \dots m \mid b \neq i, A_b \neq \Delta\} \\
 &\quad \underbrace{H(V_i | \underline{Y}, \underline{V}_S = v_S)}_{\sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j, i \notin \mathcal{S}} H(V_i | \underline{Y} \underline{V}_S)} \quad 0 \leq j = |\mathcal{S}| \leq m-1 \\
 I_E &= \frac{1}{m} \sum_{i=1}^m \left[ 1 - \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_S) \right] \\
 I_E &= 1 - \frac{1}{m} \sum_{i=1}^m \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_S)
 \end{aligned}$$

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## Derivation of Area Property 2



$$\begin{aligned}
 I_E &= 1 - \frac{1}{m} \sum_{i=1}^m \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_S) \\
 \mathcal{A} &= \int_0^1 1 - \frac{1}{m} \sum_{i=1}^m \sum_{j=0}^{m-1} (1-p)^j \cdot p^{m-1-j} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_S) dp \\
 \mathcal{A} &= 1 - \frac{1}{m} \sum_{i=1}^m \sum_{j=0}^{m-1} \int_0^1 (1-p)^j \cdot p^{m-1-j} dp \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_S) \\
 \mathcal{A} &= 1 - \frac{1}{m} \sum_{i=1}^m \sum_{j=0}^{m-1} \frac{1}{m \binom{m-1}{j}} \sum_{|\mathcal{S}|=j} H(V_i | \underline{Y} \underline{V}_S) \\
 &\quad \underbrace{\hspace{10em}}_{H(\underline{V} | \underline{Y})}
 \end{aligned}$$

$$\mathcal{A} = 1 - \frac{1}{m} H(\underline{V} | \underline{Y}) = 1 - \frac{1}{m} H(\underline{X} | \underline{Y})$$

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## Derivation of $H(\underline{V}|\underline{Y})$

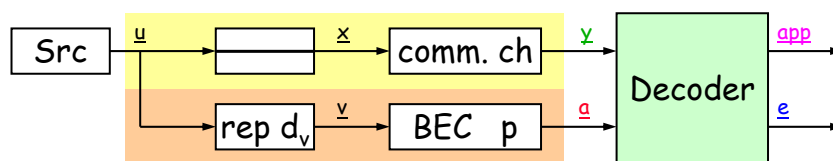


$$\begin{aligned}
 H(\underline{V}) &= H(V_1) + H(V_2|V_1) + H(V_3|V_1V_2) \\
 &= H(V_1) + H(V_3|V_1) + H(V_2|V_1V_3) \\
 &= H(V_2) + H(V_1|V_2) + H(V_3|V_1V_2) \\
 &= H(V_2) + H(V_3|V_2) + H(V_1|V_2V_3) \\
 &= H(V_3) + H(V_1|V_3) + H(V_2|V_1V_3) \\
 &= H(V_3) + H(V_2|V_3) + H(V_1|V_2V_3)
 \end{aligned}$$

$$\begin{aligned}
 3! \cdot H(\underline{V}) &= 2 \cdot (H(V_1) + H(V_2) + H(V_3)) + \\
 &+ H(V_1|V_2) + H(V_1|V_3) + H(V_2|V_1) + H(V_2|V_3) + H(V_3|V_1) + H(V_3|V_2) \\
 &+ 2 \cdot (H(V_1|V_2V_3) + H(V_2|V_1V_3) + H(V_3|V_1V_2))
 \end{aligned}$$

$$\sum_{i=1}^m \sum_{j=0}^{m-1} \frac{1}{m \binom{m-1}{j}} \sum_{|\mathcal{S}|=j} H(V_i|\underline{Y}_{\mathcal{S}})$$

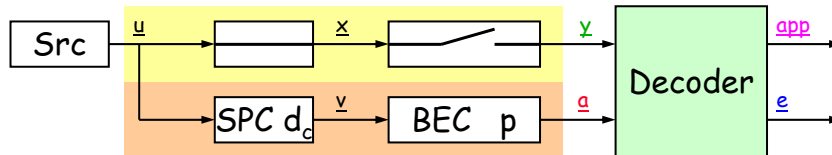
## Variable Nodes



$$\begin{aligned}
 |u| &= k \\
 |x| &= k \\
 m = |v| &= k \cdot d_v
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_v &= 1 - \frac{1}{m} H(\underline{X}|\underline{Y}) = 1 - \frac{H(\underline{X}) - I(\underline{X}; \underline{Y})}{k \cdot d_v} \\
 &= 1 - \frac{k - k \cdot I(X_1; Y_1)}{k \cdot d_v} = 1 - \frac{1 - C}{d_v}
 \end{aligned}$$

## Check Nodes



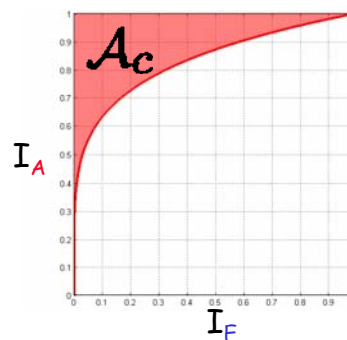
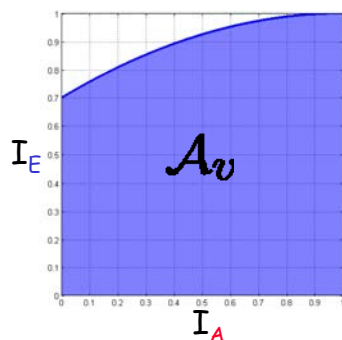
$$\begin{aligned} |u| &= k & m = |v| &= k \cdot \frac{d_c}{d_c - 1} \\ |x| &= k \end{aligned}$$

$$\begin{aligned} \mathcal{A}_c &= 1 - \frac{1}{m} H(V|Y) = 1 - \frac{H(V) - I(V; Y)}{m} \\ &= 1 - \frac{k \cdot (d_c - 1)}{k \cdot d_c} = \frac{1}{d_c} \end{aligned}$$

## Area of LDPC Component Codes

$$\mathcal{A}_v = 1 - \frac{1 - C}{d_v}$$

$$\mathcal{A}_c = \frac{1}{d_c}$$



Necessary condition for successful decoding:

$$1 - \mathcal{A}_v < \mathcal{A}_c$$

## Consequences of Area Property



$$1 - \mathcal{A}_v < \mathcal{A}_c$$

$$1 - 1 + \frac{1 - C}{d_v} < \frac{1}{d_c}$$

$$1 - C < \frac{d_v}{d_c}$$

$$C > 1 - \frac{d_v}{d_c} = R$$

**"Surprising" result:**

The area property tells us that the decoder can only converge if the rate is smaller than capacity!

## More Consequences...



Suppose the condition for convergence is fulfilled

$$1 - \mathcal{A}_v = \gamma \cdot \mathcal{A}_c < \mathcal{A}_c \quad 0 \leq \gamma < 1$$

$$\mathcal{A}_c = \frac{1}{d_c}$$

$$1 - \mathcal{A}_v = \gamma \cdot \mathcal{A}_c = \frac{1 - C}{d_v}$$

$$R = 1 - \frac{d_v}{d_c} = 1 - \frac{1 - C}{\gamma} = \frac{C - (1 - \gamma)}{\gamma} < C$$

**What is the result of this inequality?**

## Area and Rate Loss



$$R = 1 - \frac{d_v}{d_c} = 1 - \frac{1 - C}{\gamma} = \frac{C - (1 - \gamma)}{\gamma} < C$$

If  $\gamma \rightarrow 1$  we can transmit at rates that approach capacity.  
If  $\gamma < 1$  we are bounded from capacity.

$$\gamma \rightarrow 1 \text{ means that } 1 - A_v = A_c$$

Furthermore, the curves must not intersect.

$\Rightarrow$  The curves have to be matched.

Code design reduces to curve fitting!

## Curve Fitting – Code Mixture



We only considered regular codes, where every symbol has the same properties. Therefore, averaging over all symbols is equivalent to the mutual information of an arbitrarily symbol.

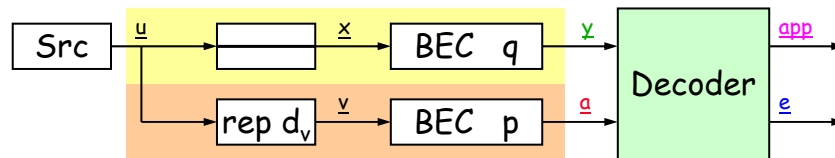
$$I_E = \frac{1}{m} \sum_{i=1}^m I(V_i; E_i) = I(V_1; E_1)$$

If we partition  $m$  into  $n_u$  groups  $j=1 \dots n_u$  each with length  $l_j$ , we can write  $I_E$  as

$$I_E = \sum_{j=1}^{n_u} \frac{l_j}{m} \left[ \frac{1}{l_j} \sum_{i=1}^{l_j} I(V_{ji}; E_{ji}) \right] = \sum_{j=1}^{n_u} \gamma_j I_{E_j} \quad \gamma_j = \frac{l_j}{m} = \frac{l_j}{\sum_{j=1}^{n_u} l_j}$$

The resulting EXIT function is the weighted average of the EXIT functions of the groups.

## Example – Variable Mixture



70% of the variable nodes have  $d_v=2$   
 30% of the variable nodes have  $d_v=5$

$$\gamma_1 = \frac{0.7 \cdot k \cdot 2}{0.7 \cdot k \cdot 2 + 0.3 \cdot k \cdot 5} = 0.48$$

$$\gamma_2 = \frac{0.3 \cdot k \cdot 5}{0.7 \cdot k \cdot 2 + 0.3 \cdot k \cdot 5} = 0.52$$

$$I_{E_j} = 1 - qp^{d_{vj}-1}$$

$$I_E = \gamma_1 \cdot [1 - qp^{d_{v1}-1}] + \gamma_2 \cdot [1 - qp^{d_{v2}-1}]$$

$$I_E(p) = 1 - q \cdot \sum_{j=1}^{n_u} \gamma_j \cdot p^{d_{vj}-1}$$

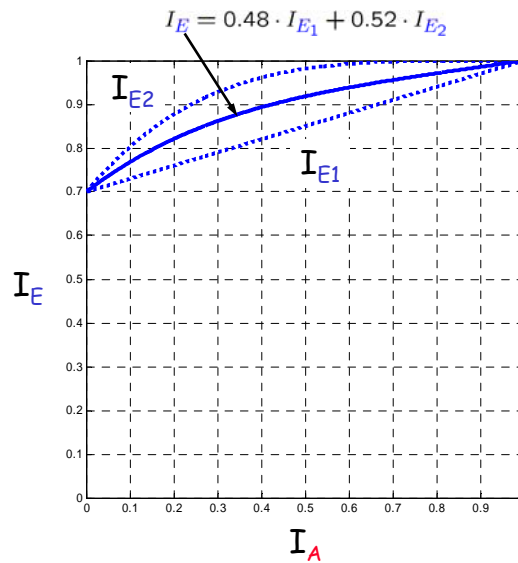
This is a polynomial in p

Note that  $\sum \gamma_j = 1$

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## Example – Variable Mixture



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## Curve Fitting



Lets fix the EXIT function of the check node decoder.

$$I_{Ec} = (I_{Ac})^{d_c-1}$$

For curve fitting, we can exchange the following quantities

$$I_{Ec} = I_{Av} \quad I_{Ev} = I_{Ac}$$

Therefore, we can write the EXIT function of the variable node decoder as the inverse EXIT function of the check node decoder.

$$I_{Av} = (I_{Ev})^{d_c-1}$$

$$I_{Ev} = (I_{Av})^{\frac{1}{d_c-1}} = (1-p)^{\frac{1}{d_c-1}}$$

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## Taylor Series Expansion



$$I_{Ev} = (I_{Av})^{\frac{1}{d_c-1}} = (1-p)^{\frac{1}{d_c-1}}$$

Assuming for example  $d_c=5$  we can expand  $I_{Ev}$  as a Taylor series

$$I_{Ev} = 1 - \left[ \frac{1}{4}p + \frac{3}{32}p^2 + \frac{7}{128}p^3 + \dots \right]$$

Truncating the Taylor series and normalizing the coefficients to 1 results in

$$I_{Ev} = 1 - \frac{51}{128} \left[ \frac{32}{51}p + \frac{12}{51}p^2 + \frac{7}{51}p^3 \right]$$

Compare this with the transfer function of the mixture of variable nodes...

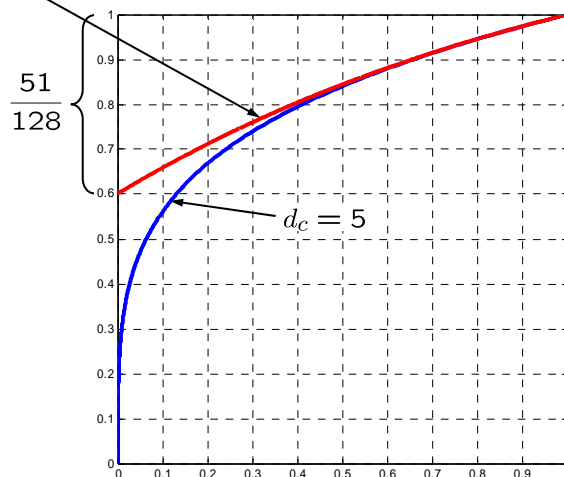
$$I_E(p) = 1 - q \cdot \sum_{j=1}^{n_u} \gamma_j \cdot p^{d_{vj}-1}$$

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## Curve Fitting



$$I_{Ev} = 1 - \frac{51}{128} \left[ \frac{32}{51}(1 - I_{Av}) + \frac{12}{51}(1 - I_{Av})^2 + \frac{7}{51}(1 - I_{Av})^3 \right]$$



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## Even more Consequences...



Using the same model as for the variable length code decoder, it can be shown that the areas for a **serial concatenated** code with an outer code  $R_{out} = k_{out}/n_{out}$  and an inner code  $R_{in} = k_{in}/n_{in}$  are given by

$$A_{out} = 1 - R_{out} \quad A_{in} = \frac{I(\underline{X}; \underline{Y})}{n_{in} \cdot R_{in}}$$

The same necessary condition  $1 - A_{out} < A_{in}$  leads to

$$R_{out} \cdot R_{in} < \frac{I(\underline{X}; \underline{Y})}{n_{in}} \leq C$$

If the inner code has rate  $< 1$ , i.e.  $I(\underline{X}; \underline{Y})/n_{in} < C$  then we can not achieve capacity with serial concatenated codes!

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## Summary



- The **area** under the information transfer function has some important consequences for the rate of the code and for **code design**.
- Capacity can only be achieved, if the **curves** of the component decoders are **matched**.
- Furthermore, **serial concatenated codes** can only achieve capacity if the rate of the **inner encoder** is  $\geq 1$ .