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EQUILIBRIUM MARKET PRICES FOR MULTI-PERIOD AUCTIONS OF INTERNET RESOURCES

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Abstract

Auctions are a well-established mechanism for efficient allocation of scarce resources and as such have already become a standard approach for pricing QoS-enabled future Internet services. This paper investigates networks where auctions are performed periodically and customers request bandwidth for sessions with holding times longer than one auction period. A simple mathematical model assuming constant session holding times, identical bandwidth requests and i.i.d. uniformly distributed bids allows to derive explicitly the resulting equilibrium market price. In the rest of the paper, this basic result is generalized by relaxing the original assumptions one after the other.

Keywords

Internet Economics, Generalized Vickrey Auction, Second-chance Auction, Nash Equilibrium

1 INTRODUCTION AND RELATED WORK

Pricing mechanisms for future QoS-enabled Internet services have received rapidly increasing attention over the last couple of years. This has led to establishing “Internet Economics” as a new and promising research area of its own, aiming at a fresh perspective on familiar problems. The basic idea of this interdisciplinary approach is to understand the Internet as an economical rather than a purely technical system, and thus to describe and solve networking issues through the use of economic concepts and techniques. While fundamental contributions to this new paradigm are collected in [12], we refer to [3] for a recent textbook introduction.

As a first consequence, the understanding of efficient resource allocation is changing. While network efficiency used to be measured by parameters like throughput or delay, the economic description is rather focused on the users and defines efficiency as “maximizing the social welfare”, expressed as sum of the users’ utilities (or “valuations”). Thus, an efficient mechanism allocates scarce resources to those users who valuate them most, and the simplest mean for finding out the latter is to charge users for what they get. Therefore, pricing schemes play *the* central role in Internet Economics, and this has led to considerable research efforts in this direction (for a general overview see [4] or [16]).

Among the many proposed pricing approaches, the adaptation of auctions to the context of Internet services seems to be especially appealing. In general, an auction is defined as a market mechanism with explicit rules determining resource allocation and prices on the basis of bids from the market participants (in the case of networks usually represented by respective user agents) [9]. Since Vickrey's seminal paper [17], the economic theory of auctions has been subject to intensive research [7]. Vickrey's ingenious discovery was that so-called "Second-price Auctions" (a.k.a. "Vickrey Auctions"), where the winning bidder does not pay her own bid but the bid of the highest-bidding loser, lead (under rather general assumptions) to the same expected outcome as any other mechanism ("Revenue Equivalence Theorem"), but additionally force the competitors to be honest about their true valuations of the auctioned commodity ("incentive compatibility"). MacKie-Mason and Varian [11] were the first to propose a second-price auction scheme for charging packet transmission, but their "smart market" approach determining an individual market price for each single packet lacks scalability. Therefore, the original concept has been adapted to the case of flows [10] and arbitrarily divisible capacities, resulting in the "Progressive Second Price" (PSP) mechanism [8]. Here, the total charge to be paid is no longer uniform for each winner, instead the charge is calculated as a weighted mix of the bids submitted by those losing competitors who are displaced by the specific winning user, with weights corresponding to the losers' demands (i.e. large losing demands have big influence on the price).

Recently, auctions have been also investigated in the context of routing [5], scheduling [18], provisioning of MPLS networks [1], and particularly for bandwidth allocation over network paths [2, 6, 14]. [14] for the first time has studied an auction scenario for multi-period sessions over network paths. The resulting CHiPS (Connection-Holder-is-Prefered Scheme) mechanism is designed to the advantage of users who have already established an end-to-end connection and experience an unexpected local market perturbation on one of the links along their connection. Instead of shutting down the global connection because of such a local event, the users are given a second-chance for submitting a sufficient bid ex posteriori. CHiPS is complemented by MIDAS (Multilink Dutch Auction Scheme) due to Courcoubetis et al. [2] who describe a descending-bid second-price auction for establishing an end-to-end connection over several links with different congestion situations. Finally, [13] and [19] investigate multi-period auctions in more detail and propose several flavors of the SAM (Second-chance Auction Mechanism) scheme. In its simplest version, each auction is restricted to the part of the link bandwidth which is currently free, and distributes this free bandwidth among newly arriving competitors, at the same time determining the current market price. Already established sessions are allowed to continue, but may be required to adapt their current bid ex post to the actual market price.

The present paper focuses to the latter mechanism and provides an analytical derivation of the resulting equilibrium market prices for multi-period auctions. Section 2 presents a simple model for the case of users with identical bandwidth requests and session holding times as well as a uniform distribution of the submitted bids. After deriving the equilibrium auction result for this case, the rest of the paper generalizes the simplifying assumptions one after the other: section 3 deals with the equilibrium auction result if the session holding time may differ between users, section 4 investigates the highly complicated case of different bandwidth requests per user, and section 5 considers general bid distributions. Section 6 concludes the paper with summary and outlook. Note that due to space restrictions, in this paper most proofs are only sketched; we refer to [15] for further details.

2 A Simple Model for Multi-Period Second-Chance Auctions

We consider a single link with capacity *C*, where for reasons of simplicity we assume *C* to be a natural number of channels. The auction mechanisms applied in our model is the Generalized Vickrey Auction (GVA) and the Progressive Second-Price Auction (PSP), resp. Beyond the general assumption that all random variables are considered to be mutually independent, we make the following additional assumptions:

Assumption 1: *In each auction period, new users arrive as a Poisson stream of rate* ν. *An arriving user i submits a (demand, price)-pair as bid, where* β_i *describes the price user i is willing to pay for each of the* d_i *bandwidth units she is requesting.*

Assumption 2: *Each user i has identical demand* $d_i \equiv d = 1$*, i.e. each user requests one channel for her session holding time.*

Assumption 3: *Each user i has identical session holding time* $\sigma_i \equiv T$*, i.e. each user needs to win the same number of subsequent auctions.*

Assumption 4: *User valuations (and hence user bids* β_i *under GVA/PSP) are uniformly* $U[a;b]$ *distributed. Thus, with probability 1 no two bids are identical.*

Additionally, we define the following terminology:
- "*winning bid*" / "*losing bid*"= bid that can be fulfilled completely / cannot be fulfilled even partially.
- "*lowest winning bid*" / "*highest losing bid*" = the unique bid that has the lowest price component among all winning bids / highest price component among all losing bids.
- "*border winning bid*" = "*border losing bid*" = a bid whose price component is smaller or equal to all winning bids and at the same time larger or equal to all losing bids.
- "*price-relevant bid*" = bid that contributes under PSP to the charge calculation for related winning bids.

Remark 0: As described in [19], the characterization of second-chance auction (SAM) schemes includes user *i* having a total budget of B_i for her session of length σ_i, splitting B_i among the individual bids $\beta_i(t)$ for auction *t*, $t = 1, \ldots, \sigma_i$, and potentially correcting $\beta_i(t)$ for $t \geq 2$ ex-post. It is important to note that in our equilibrium case, SAM coincides with the above model (with $\beta_i(t) \equiv \beta_i = B_i / \sigma_i \ \ \forall t$) because the market price does not change in time and therefore rebidding is not necessary as soon as a session is established.

Theorem 2.1 (Equilibrium Price for Multiperiod GVA):

Under assumptions 1, 2, 3 and 4, the expected equilibrium price π *of a Generalized Vickrey Auction (GVA) is*

$$\pi = a + \left(\nu - \frac{C}{T}\right) \cdot \frac{b-a}{\nu+1}. \tag{1}$$

Proof:

This proof is based on standard order statistics [21]: Let $\{X_n, n = 1, \ldots, N\}$ be a set of *N* i.i.d. (independent identically distributed) uniformly $U[a, b]$-distributed random variables. Then $\mathbb{E}(X_{(n)})$, the expectation value of the *n*-th lowest value, can be described as

$$\mathbb{E}(X_{(n)}) = a + \frac{n}{N+1}(b-a). \tag{2}$$

Assume now π to be the expected result of a second-chance auction (SAM) in a Nash equilibrium. Then each of the ω "old" users (which have survived already at least one auction and still participate at the present auction) has a valuation larger or equal to π (due to the equilibrium assumption). Hence the valuations of the old users are uniformly $U[\pi, b]$-distributed. The highest losing bid price, which defines the auction outcome in a GVA, will certainly come from one of the $L = \nu + \omega - C$ losing new users. Using (2), we get

$$\pi = a + L \cdot \frac{b-a}{\nu+1} = a + (\nu + \omega - C) \cdot \frac{b-a}{\nu+1}. \quad (3)$$

To derive the equilibrium number ω of old users, define ω_t to be the number of old users which have already won exactly t auctions. As all users are assumed to have identical session holding times, in equilibrium $\omega_1 = \omega_2 = \ldots = \omega_T = \frac{C}{T}$, and thus

$$\omega = \sum_{t=1}^{T-1} \omega_t = \frac{T-1}{T} \cdot C \quad (4)$$

Inserting (4) into (3) yields eventually the desired result (1). ❑

Thus, we have proved the central result (1) for the equilibrium market price for bidders having identical bandwidth requests, identical session holding times and uniformly distributed bids. Now we will generalize Theorem 2.1 by relaxing these assumptions one by one.

3 Model Extension I: Varying Session Lengths

For dealing with the case of varying session lengths we relax Assumption 3 as follows:

Assumption 3*: *Session holding times σ_i are no longer identical, but uniformly distributed over the discrete set $\{T, T+1, \ldots, T+\kappa\}$.*

Theorem 3.1 (Equilibrium Price for Varying Session Holding Times):

Under assumptions 1, 2, 3 and 4, the equilibrium market price π for a PSP equals*

$$\pi = a + \left(\nu - \frac{2C}{2T+\kappa}\right) \cdot \frac{b-a}{\nu+1}. \quad (5)$$

Sketch of Proof:

The proof is parallel to the proof of Theorem 2.1, except for the fact that we have to derive a more complicated expression for the equilibrium number ω of old users. Therefore, let $\omega_{s,t}$ be the number of old users with session length s having won already $1 \le t \le s$ consecutive auctions. Then, the uniform distribution of holding times together with the equilibrium assumption yields $\forall\ s, s' = T, \ldots, T+\kappa;\ \ \forall\ t, t' = 1, \ldots, s$:

$$\omega_{s,t} = \omega_{s,t'} = \omega_{s',t'}, \quad (6)$$

i.e. all classes $\omega_{s,t}$ of old users with session length s and t previously won auctions have equal size. Moreover, the total number of old user classes $\omega_{s,t}$ equals

$$\sum_{s=1}^{T} (\kappa+1) + \sum_{s=T+1}^{T+\kappa} (s-T) = T(\kappa+1) + \frac{\kappa(\kappa+1)}{2} = \frac{(2T+\kappa)(\kappa+1)}{2}. \quad (7)$$

All old users with $s = t$ are finishing their sessions (i.e. a total of $\kappa + 1$ classes). Hence, the expected number of old users who continue their bidding equals

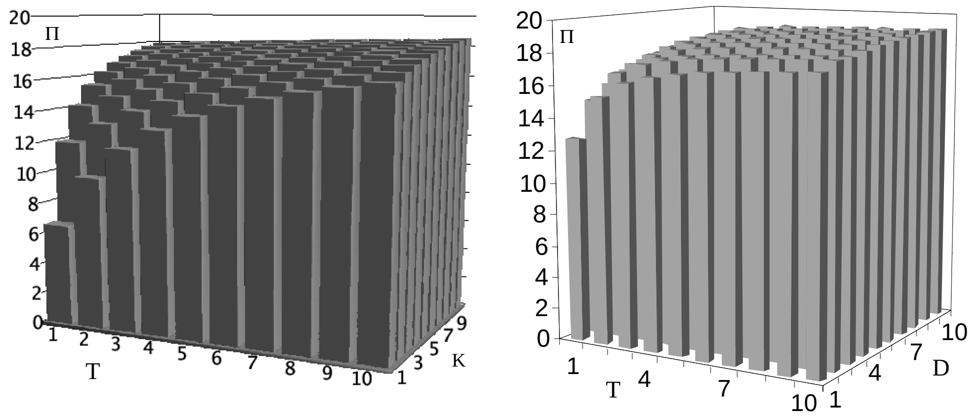

Figure 1: Asymptotic Behaviour of Equilibrium Prices: Varying Session Holding Times (left) and Varying Demands (right)

$$\omega = C \cdot \frac{(2T+\kappa)(\kappa+1)-2(\kappa+1)}{(2T+\kappa)(\kappa+1)} = C \cdot \frac{2T+\kappa-2}{2T+\kappa} = \left(1-\frac{2}{2T+\kappa}\right)C. \tag{8}$$

Using this result for equation (3), this finally leads to

$$\pi = a + (\nu + \omega - C) \cdot \frac{b-a}{\nu+1} = a + \left(\nu - \frac{2}{2T+\kappa}C\right) \cdot \frac{b-a}{\nu+1}. \qquad \square \tag{9}$$

Remark 1: For $\kappa = 0$ (case of assumption 3), (9) reduces to $\pi = a + \left(\nu - \frac{C}{T}\right)\frac{b-a}{\nu+1}$ which is consistent to Theorem 2.1.

Figure 1 (left) illustrates the equilibrium auction results for a scenario with $C = 100$, $\nu = 100$, $a = 0$, $b = 20$ and varying parameters T and κ. The resulting function is monotonously increasing in T as well as κ. Note that in both limiting regimes $T \to \infty$ and $\kappa \to \infty$, the equilibrium price converges towards $\pi \to a + \nu \cdot (b-a)/(\nu+1)$, and for $\nu \to \infty$, this expression converges towards b, i.e. the upper bound of the interval of possible bids. This has a straightforward interpretation: the longer the sessions are, the fewer new users can start their sessions, and the market price increases. And the more new users there are, the higher is the price required to get one of the few free places. Therefore, our results show a perfectly consistent behaviour of the equilibrium market prices.

4 Model Extension II: Varying User Demands

As a second generalization step, we allow users to have different bandwidth requests, while they are submitting now bids β_i per bandwidth unit.

Assumption 2*: *User demand is uniformly distributed over the discrete set* $\{1, 2, \ldots, D\}$.

The resulting equilibrium market price is described in Theorem 4.4. The proof is complex, because a winning bidder with large bandwidth request may throw out several competitors with differing requests and bids, and also the possibility of a bidder who can only partially be satisfied has to be taken into account properly. We start with some preparing lemmata.

Lemma 4.1:
Define a border winning bidder to have the 0-th losing bid. Then, under assumptions 1, 2, 3 and 4, and with* ω' *as average number of old users in analogy to (4), in equilibrium the expected value* $b_{(k)}$ *of the k-th losing bid equals*

$$b_{(k)} = a + \left(\nu + \omega - \frac{2C}{D+1} - (k-1)\right) \cdot \frac{b-a}{\nu+1}. \tag{10}$$

Sketch of Proof:
First, the expected demand of a user (independently of whether being an old or a new one) is calculated and the average number of old users continuing their sessions as introduced in (4) is adapted. Again, in equilibrium, no old user will lose the auction, therefore the number of losing bidders equals $L = \nu - \frac{2C}{(D+1)T}$. Applying the usual order statistics yields (10). ❑

Lemma 4.2:
Under assumptions 1, 2, 3 and 4, define* q_s *as the probability that the border winning bid receives s units of bandwidth less than requested. Then,*

$$q_s = \sum_{m=m_0}^{C} \frac{\xi_m^{(D)}(s)}{D^m} \qquad \forall s = 0, \ldots, D-1, \tag{11}$$

where $m_0 = \lceil C/D \rceil$ *and* $\xi_m^{(D)}(s)$ *describes the number of vectors with length m whose components are not larger than D and altogether sum up to s.*

Sketch of Proof:
Define $\Delta_m(s)$ to be the set containing all possible demand constellation vectors δ of length *m*, where each individual demand may range between 1 and *D* channels, with *m* being any number between $m_0 = \lceil C/D \rceil$ and *C*. In order to calculate q_s, we consider a vector of *C* users, each of which can have a demand between 1 and *D* channels, as the statistical universe (i.e. the number of possible cases). The number of opportune cases if a total of *m* bids are accepted, and the border winning bid gets *s* channels less its request, is equal to $|\Delta_m(s)| \cdot D^{C-m}$, as the request size does not matter for the losing $C-m$ bids. Calculating the ratio between opportune and possible cases yields (11). ❑

Coming back to our original problem, as next step consider a situation where the border winning bid receives *s* units less than requested, $0 \le s \le D-1$. Moreover, consider a winning bid, whose complete demand of $l > 1$ can be fulfilled, and which causes *k* competing bids to become losing bids or a border losing bid, resp., $1 \le k \le l$, and define $\psi_{s,k}^{(l)}$ to be a *k*-dimensional vector of price-relevant demands (in falling order of the price components):

$$\psi_{s,k}^{(l)} \in \Psi_{s,k}^{(l)} = \left\{ (\tilde{d}_1^{(l)}, \ldots, \tilde{d}_k^{(l)}) \;\middle|\; \sum_{n=1}^{k-1} \tilde{d}_n^{(l)} < l \le \sum_{n=1}^{k} \tilde{d}_n^{(l)}, \tilde{d}_n^{(l)} \ge 1 \right\}. \tag{12}$$

$\Psi_{s,k}^{(l)}$ is further structured by subsuming all elements that differ only w.r.t. their last component $\tilde{d}_k^{(l)}$ into a single class $(\tilde{d}_1^{(l)}, \ldots, \tilde{d}_{k-1}^{(l)}, \cdot) = \bar{\psi}_{s,k}^{(l)}$, which represents a total number of

$$\left|\bar{\psi}_{s,k}^{(l)}\right| = D - l + \sum_{n=1}^{k-1} \tilde{d}_n^{(l)} + 1 \tag{13}$$

elements of $\Psi_{s,k}^{(l)}$. The following Lemma describes the probability $r(\bar{\psi}_{s,k}^{(l)})$ of this class with respect to all possible demand constellations of *k* price-relevant bids for given *l* and *s*.

Lemma 4.3:

Consider the set

$$\bar{\Psi}_{s,k}^{(l)} = \left\{ (\underset{\sim}{d}_1^{(l)}, \ldots, \underset{\sim}{d}_k^{(l)}) \,\middle|\, \sum_{n=1}^{k} \underset{\sim}{d}_n^{(l)} = l, \underset{\sim}{d}_n^{(l)} \geq 1 \right\}. \tag{14}$$

There is a one-to-one correspondence between each element of $\Psi_{s,k}^{(l)}$ *and one of the classes* $\bar{\psi}_{s,k}^{(l)}$. *Moreover,*

$$r(\bar{\psi}_{s,k}^{(l)}) = \frac{D - l + 1 + \sum_{n=1}^{k-1} \underset{\sim}{d}_n^{(l)}}{D^k} (I\!I_{[s=0]} + D \cdot I\!I_{[s>0]}). \tag{15}$$

Sketch of Proof:

After proving the one-to-one relationship, two cases have to be distinguished: If $s > 0$, then the size of the statistical universe equals D^{k-1}, whereas for $s = 0$, the size of the statistical universe equals D^k. This yields together with (13) the assertion. ❑

Theorem 4.4 (Equilibrium Market Price for Varying User Demand):

Under assumptions 1, 2, 3 and 4, and given the definitions of the previous Lemmas, the equilibrium market price* π *per unit of a winning bid equals*

$$\pi = \frac{1}{D} \sum_{l=1}^{D} \left[\frac{1}{l} \sum_{s=0}^{D-1} (q_s \Phi_s(l)) \right], \tag{16}$$

where

$$\Phi_s(l) = \begin{cases} l \cdot b_{(0)} & \text{if } s \geq l \geq 1 \\ \sum_{k=1}^{l} \left[\sum_{\bar{\psi}_{0,k}^{(l)} \in \bar{\Psi}_{0,k}^{(l)}} \left(r(\bar{\psi}_{0,k}^{(l)}) \sum_{n=1}^{k} \underset{\sim}{d}_n^{(l)} \cdot b_{(n)} \right) \right] & \text{for } s = 0 \\ \sum_{k=1}^{l} \left[\sum_{\bar{\psi}_{0,k}^{(l)} \in \bar{\Psi}_{0,k}^{(l)}} \left(r(\bar{\psi}_{0,k}^{(l)}) \sum_{n=1}^{k} \underset{\sim}{d}_n^{(l)} \cdot b_{(n-1)} \right) \right] & \text{otherwise} \end{cases}. \tag{17}$$

Sketch of Proof:

Again, we consider first a winning bid receiving its complete bandwidth request of $l \geq 1$. Following the standard PSP charging rule [8], the expected total charge for a winning bid is calculated as weighted average of those losing bids or border losing bids, resp., that are thrown out by the winning bid. The resulting charge is described by (17) for each of the three cases $s \geq l$, $s = 0$ and $1 \leq s < l$. In order to calculate the expected market price, this term has still to be averaged over *l* and *s*, thus yielding the final expression (16). ❑

Figure 1 (right) illustrates the resulting equilibrium prices for varying parameters *D* and *T*. Similarly to Figure 1 (left), we observe a convergent behaviour for large demands and session holding times.

Remark 2: Consider the case $D = 1$ corresponding to assumption 2. In this case, the border winning bid receives its complete demand of $l = 1$, hence $s = 0$ and $q_0 = 1$ according to Lemma 4.2. Thus, (16) reduces to $\pi = \Phi_0(1)$. Moreover, there is only one price-relevant demand, hence the vector $\psi_{s,k}^{(l)}$ as defined in (12) reduces to the scalar $\tilde{d}_1^{(1)} = 1$. Therefore, with (15), the corresponding probability $r(\bar{\Psi}_{0,1}^{(1)})$ becomes 1.

Putting this together into (17), $\Phi_0(1) = \sum_{\bar{\Psi}_{0,1}^{(1)} \in \bar{\Psi}_{0,1}^{(1)}} r(\bar{\Psi}_{0,1}^{(1)}) \cdot \tilde{d}_1^{(1)} \cdot b_{(1)} = b_{(1)}$.

As $b_{(1)}$ is calculated already in Lemma 4.1, we end up again with Theorem 2.1.:

$$\pi = \Phi_0(1) = b_{(1)} = a + (\nu + \omega - C) \cdot \frac{b-a}{\nu + 1}. \tag{18}$$

5 Model Extension III: General Bid Distributions

Finally, we also generalize the assumption that the (unit) bids are uniformly distributed. The uniform distribution is a nice case for analytical purposes, as it allows (2) as a simple explicit expression for the expected value of the *n*-th order statistics. For more general bid distributions, the approach of Theorem 2.1 remains unchanged except for a more general description of the *n*-th order statistics.

Assumption 4*: *User bids β_i are i.i.d. random variables with an arbitrary probability distribution P where the inverse function P^{-1} exists.*

Lemma 5.1:
Assume $X_1, X_2, \ldots, X_N$ to be N i.i.d. random variables with probability distribution $P(x)$. Generalizing (2), the expectation value of the n-th order statistics (i.e. the expectation of the n-th lowest realized value) is

$$x_{(n)} = P^{-1}\left(\frac{n}{N+1}\right). \tag{19}$$

Proof by complete induction:

For $N = 1$, the expected value $x_{(1)}$ of the sole realization is obviously placed at the 50% quantile, where $P(x_{(1)}) = 1/2$.

Now assume that the assertion is valid for a natural number $N \geq 1$. Consider *N*+1 i.i.d. random variables $X_1, X_2, \ldots, X_{N+1}$ and let $x_{(1)}$ be the lowest realized value. Then, we may define a new cumulative probability function $P'(x)$ for $x \geq x_{(1)}$ as follows:

$$P'(x) = \frac{P(x) - P(x_{(1)})}{1 - P(x_{(1)})}. \tag{20}$$

Applying the induction assertion to $P'(x)$ and the remaining *N* random variables we get $P'(x_{(n+1)}) = \frac{n}{N+1} = \frac{P(x_{(n+1)}) - P(x_{(1)})}{1 - P(x_{(1)})}$, and hence

$$P(x_{(n+1)}) = (1 - P(x_{(1)}))\frac{n}{N+1} + P(x_{(1)}) = \frac{n}{N+1} + \left(1 - \frac{n}{N+1}\right)P(x_{(1)}),\ n \geq 1\,. \quad (21)$$

Similarly, we now assume $x_{(N+1)}$ to be the largest of the N+1 realizations and consider the remaining N ones. This gives us as new probability distribution $P''(x)$ for $x < x_{(N+1)}$:

$$P''(x) = \frac{P(x)}{P(x_{(N+1)})}\,. \quad (22)$$

Applying again the induction assertion to $P''(x)$ and the remaining N random variables we get $P''(x_{(n)}) = \frac{n}{N+1} = \frac{P(x_{(n)})}{P(x_{(N+1)})}$ and hence

$$P(x_{(n)}) = \frac{n}{N+1}(P(x_{(N+1)})) \text{ for } n \leq N\,. \quad (23)$$

For $x_{(1)}$ we get $P(x_{(1)}) = \frac{1}{N+1}(P(x_{(N+1)})) = \frac{1}{N+1}\left(\frac{N}{N+1} + \frac{1}{N+1}P(x_{(1)})\right)$ from combining (23) and (21), hence $\left(1 - \frac{1}{(N+1)^2}\right)P(x_{(1)}) = \frac{N}{(N+1)^2}$, and therefore

$$P(x_{(1)}) = \frac{N}{(N+1)^2 - 1} = \frac{1}{N+2}\,. \quad (24)$$

Then, (21) yields $P(x_{(N+1)}) = \frac{N}{N+1} + \left(\frac{1}{N+1}\right)\frac{1}{N+2} = \frac{N(N+2)+1}{(N+1)(N+2)} = \frac{N+1}{N+2}$ (25)

and finally combining (23) and (25) we get

$$P(x_{(n)}) = \frac{n}{N+1} \cdot \frac{N+1}{N+2} = \frac{n}{N+2} \text{ for } 1 < n < N+1\,. \quad (26)$$

Thus, (24), (25) and (26) together yield the assertion for N+1. ❑

Theorem 5.2 (Equilibrium Market Price for Arbitrary Bid Distributions):
Under assumptions 1, 2, 3 and 4, for a bid distribution characterized by the (invertible) cumulative distribution function P, the equilibrium market price equals*

$$\pi = P^{-1}\left(\frac{\nu - C/T}{\nu + 1}\right). \quad (27)$$

Sketch of Proof:
The proof is identical to the proof of Theorem 2.1, except for the fact that we use now the generalized order statistics of Lemma 5.1. ❑

Remark 3: For the uniform $U[a;b]$ bid distribution (Assumption 4), $P(x) = \frac{x-a}{b-a}$ for $a \leq x \leq b$, hence $P^{-1}(x) = a + x(b-a)$ for $0 \leq x \leq 1$, and therefore (27) yields $\pi = P^{-1}\left(\frac{\nu - C/T}{\nu + 1}\right) = a + (b-a)\frac{\nu - C/T}{\nu + 1}$ which again is consistent to Theorem 2.1.

6 SUMMARY AND CONCLUSIONS

This paper provides an analytical investigation of the equilibrium market prices resulting from multiperiod auctions on a single link. We have started from a simplified model and subsequently relaxed the basic assumptions one after the other. Detailed proofs have been provided for a couple of theorems containing the corresponding generalized equilibrium results. The consistency with the simple model is demonstrated, and some illustrations are given. Current and future work considers a generalization of the distributions for request sizes and session holding times.

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